

The “universal property” of horizon entropy sum of black holes in four dimensional asymptotical (anti-)de-Sitter spacetime background

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ABSTRACT: We present a new “universal property” of entropy, that is the “entropy sum” relation of black holes in four dimensional (anti-)de-Sitter asymptotical background. They depend only on the cosmological constant with the necessary effect of the un-physical “virtual” horizon included in the spacetime where only the cosmological constant, mass of black hole, rotation parameter and Maxwell field exist. When there is more extra matter field in the spacetime, one will find the “entropy sum” is also dependent of the strength of these extra matter field. For both cases, we conclude that the “entropy sum” does not depend on the conserved charges M , Q and J , while it does depend on the property of background spacetime. We will mainly test the “entropy sum” relation in static, stationary black hole and some black hole with extra matter source (scalar hair and higher curvature) in the asymptotical (anti-)de-sitter spacetime background. Besides, we point out a newly found counter example of the mass independence of the “entropy product” relation in the spacetime with extra scalar hair case, while the “entropy sum” relation still holds. These result are indeed suggestive to some underlying microscopic mechanism. Moreover, the cosmological constant and extra matter field dependence of the “entropy sum” of all horizon seems to reveal that “entropy sum” is more general as it is only related to the background field. For the case of asymptotical flat spacetime without any matter source, we give a note for the Kerr black hole case in appendix. One will find only mass dependence of “entropy sum” appears. It makes us believe that, considering the dependence of “entropy sum”, the mass background field may be regarded as the next order of cosmological constant background field and extra matter field. However, fully explaining the relationship between the “entropy sum” relation and background properties still requires further exploration.

KEYWORDS: Black Holes, Classical Theories of Gravity

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1 Introduction

Understanding the origin of black hole entropy at the microscopic level has been a major challenge in quantum theories of gravity in the past years. More recent interests have been focused on the “area product” or “entropy product” of the black holes which possess more than one horizon [1–16]. It seems clear that this additional thermodynamic relation of entropy appears to be “universal” and may provide further insight into the quantum physics of black holes. Various black holes include Einstein-Maxwell gravity and in Supergravity models are tested [1–7, 14]. The product of entropy is once expected to be more universal and in fact independent of the mass of the black hole [1–12]. However, It fails in some cases [13–16]. For example, in discussing the Schwarzschild-de Sitter black hole and Reissner-Nordstrom-anti-de Sitter black hole in $3 + 1$ dimensions, it has been shown that the product of event horizon area and cosmological horizon area is not mass independent, even if including the effect of the third un-physical “virtual” horizon the result does not improve [14]. This mass-dependence of the product of physical horizon areas is soon discussed more clearly in the higher curvature gravity models [16].

Now people expect more general additional thermodynamic relation. It is the primary work of this paper. In our present research, we find another “universal property” of entropy, the “entropy sum” relation of black holes in four dimensional (anti-)de-Sitter asymptotical background. They depend only on the cosmological constant with including the necessary effect of the un-physical “virtual” horizon in the spacetime where only cosmological

constant, mass of black hole, rotation and Maxwell field exist. When there is more extra matter field in the spacetime, one will find the “entropy sum” is also dependent of the strength of these extra matter field. For both cases, we conclude that the new “universal property”, that is, the “entropy sum” relation does not depend on the conserved charges M (mass), Q (charge from Maxwell field) and J (from rotation case), while it does depend on the property of background spacetime. To express it more accurately, it does depend on those constants, which characterize the strength of the background fields. We will mainly test the “entropy sum” relation in static, stationary black hole and some black hole with other extra matter source (scalar hair and higher curvature terms) in asymptotical (anti-)de-sitter spacetime background. Besides, we point out a newly discovered failed example of the mass independence of the “entropy product” relation in the discussion about the spacetime with scalar hair, while the “entropy sum” relation still holds. These result are indeed suggestive of some underlying microscopically. Anyway, the cosmological constant and extra matter field dependence of the “entropy sum” relation of all horizon seems to reveal that “entropy sum” is more general and is only related to the background field. One may be curious about the “entropy sum” in asymptotical flat spacetime without any matter source. We will give a note of Kerr black hole case in appendix B, in which one will find only mass dependence of “entropy sum” appears. It makes us believe that, considering the dependence relation of the “entropy sum”, the mass background field may be regarded as the next order of cosmological constant background field and extra matter field, while the Maxwell field and “rotation field” always play no role. Explaining the relationship between the “entropy sum” and background spacetime properties still are open problems and left to be a future work.

The surprising discovery of the cosmic late stage accelerating expansion has inspired intensive research on the universe background cosmological constant problem, including its functioning in astrophysics. The present paper is organized as follows: in next section, we test the “entropy sum” in (A)dS black hole without other extra matter field, including the Schwarzschild-de-Sitter solution, Reissner-Nordstrom-de-Sitter solution and the Kerr-(anti-)de-Sitter solution; section 3 is devoted to the discussion of “entropy sum” of the black holes with extra scalar hair and the charged rotating and static black holes in Einstein-Weyl theory. We derive the “entropy sum” and the dependence of background field constant for each of these black hole solutions. In the end of the paper, we make some conclusion and discussion.

2 “Entropy sum” of (A)dS black hole with charge and rotation

In this section, we test the “Entropy sum” of (A)dS black hole without other extra matter field, including Schwarzschild-de-Sitter, Reissner-Nordstrom-de-Sitter Solution and Kerr-(anti-)de-Sitter solution in four dimensions. Here are only cosmological constant, mass of black hole, rotation and Maxwell field in the spacetime. When there is no other extra matter fields in the (A)dS spacetime, one will find the “entropy sum” of black holes depend only on the cosmological constant with the necessary effect of the un-physical “virtual” horizon included. The Maxwell field and “rotation field” always play no role.

2.1 Warm-up: Schwarzschild-de-Sitter and Reissner-Nordstrom-de-Sitter solution

To give a warm-up, we begin the discussion with the simple static and uncharged example, four dimensional Schwarzschild-de-Sitter solution which is behaviours as

$$ds^2 = -\Delta(r)dt^2 + \frac{dr^2}{\Delta(r)} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \quad (2.1)$$

where M is the mass of the black hole and Λ is the cosmological constant, and the horizon function is

$$\Delta(r) = 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3} \quad (2.2)$$

We will substitute $\Lambda = \frac{1}{L^2}$ for convenience in this subsection. As we are aim to the “entropy sum” of black hole horizons, we first list the three roots of $\Delta(r)$ [14]

$$\begin{aligned} r_1 &= 2L \sin \left(\frac{1}{3} \arcsin \left(\frac{3M}{L} \right) \right) \\ r_2 &= 2L \sin \left(\frac{1}{3} \arcsin \left(\frac{3M}{L} \right) + \frac{2\pi}{3} \right) \\ r_3 &= 2L \sin \left(\frac{1}{3} \arcsin \left(\frac{3M}{L} \right) - \frac{2\pi}{3} \right) \end{aligned}$$

where, r_1 represents an event horizon and r_2 is a cosmological horizon. Both are physical horizons. The third one r_3 , however, is not a physical, said to be a “virtual” horizon. It is easy to find the product of event horizon area and cosmological horizon area is not mass independent, even including the effect of the third un-physical “virtual” horizon does not improve the result [14].

On the other hand, there is an exact result that

$$\sum_{i=1}^3 r_i^2 = 6L^2, \quad (2.3)$$

which immediately deriving the “area sum” of all horizons as

$$\sum_{i=1}^3 A_i = 24\pi L^2 = \frac{24\pi}{\Lambda}. \quad (2.4)$$

We note that the sum of the areas is a constant directly related to the spacetime background i.e. cosmological constant. In fact, for Schwarzschild-de-Sitter black hole, the entropy of each horizon (include the “virtual” one) is $S_i = A_i/4 = \pi r_i^2$. So we conclude

$$\sum_{i=1}^3 S_i = \frac{6\pi}{\Lambda}, \quad (2.5)$$

the “entropy sum” of all four horizons is only cosmological constant dependence and also mass independence.

However, there is no conserved charges in the Schwarzschild-de-Sitter spacetime. To be more convictive, we present the Reissner-Nordstrom-de-Sitter solution as a second warm-up. The Reissner-Nordstrom-de-Sitter solution is

$$ds^2 = -\Delta(r)dt^2 + \frac{dr^2}{\Delta(r)} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \quad (2.6)$$

where

$$\Delta(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3}. \quad (2.7)$$

In principle the quartic can be solved explicitly, but here it is not necessary to list the roots. This argument is shown in detail in [14] that there are four physical roots: each of them stand for the event horizon, Cauchy horizon, cosmological horizon and an un-physical virtual horizon respectively. The mass independence of “entropy product” of all horizons still hold [14].

For our interest, the “area sum” of all four horizons is

$$\begin{aligned} \sum_{i=1}^4 r_i^2 &= (r_1 + r_2 + r_3 + r_4)^2 - 2 \sum_{i<j} r_i r_j = 6L^2 \\ \sum_{i=1}^4 A_i &= 24\pi L^2 \end{aligned} \quad (2.8)$$

Again,

$$\sum_{i=1}^4 S_i = \frac{6\pi}{\Lambda}, \quad (2.9)$$

the “entropy sum” of all four horizons is only cosmological constant dependence and also does not depend on the conserve charges: mass M and charge Q . Namely, Mass and the Maxwell field do no effect on the “entropy sum”.

2.2 Kerr-(anti)-de-Sitter black holes

We continue our discussion with cosmological constant, mass and angular momentum of black hole exist in the spacetime, i.e. the familiar Kerr-(anti-)de-Sitter black hole [17–19]

$$ds^2 = -\frac{\Delta}{\rho^2} \left(dt - \frac{a \sin^2 \theta d\varphi}{\Xi} \right)^2 + \frac{\rho^2 dr^2}{\Delta} + \frac{\rho^2 d\theta^2}{\Delta_\theta} + \frac{\Delta_\theta \sin^2 \theta}{\rho^2} \left(a dt - \frac{(a^2 + r^2) d\varphi}{\Xi} \right)^2,$$

where

$$\begin{aligned} \rho^2 &= a^2 + r^2 \cos^2 \theta & \Delta &= (a^2 + r^2) \left(1 \mp \frac{r^2}{l^2} \right) - 2mr \\ \Delta_\theta &= 1 \pm \frac{a^2 \cos^2 \theta}{l^2} & \Xi &= 1 \pm \frac{a^2}{l^2} \end{aligned}$$

and the cosmological constant $\Lambda = \pm \frac{3}{l^2}$. Here, the upper and lower of sign stand for the dS and AdS solution respectively. The four roots of Δ is shown in [17]. They satisfy the following equality

$$\sum_{i=1}^4 r_i^2 = 2l^2 - 2a^2, \quad \text{for dS spacetime;} \quad (2.10)$$

$$\sum_{i=1}^4 r_i^2 = -2l^2 - 2a^2, \quad \text{for AdS spacetime;} \quad (2.11)$$

It is well known that the area for each horizon is

$$A(r_i) = \frac{4\pi(r_i^2 + a^2)}{1 \pm \frac{a^2}{l^2}} \quad (2.12)$$

Thus we obtain the “area sum” of all four horizons

$$\sum_{i=1}^4 A(r_i) = \pm 8\pi l^2 = \frac{24\pi}{\Lambda} \quad (2.13)$$

with the “entropy sum”

$$\sum_{i=1}^4 S(r_i) = \frac{6\pi}{\Lambda}. \quad (2.14)$$

Then we demonstrate that, in four dimensional (A)dS spacetime, the cosmological constant dependence of “entropy sum” of all horizons is a universal property. The “entropy sum” is the constant, which is proportional to cosmological radius and inversely proportional to cosmological constant, no matter charge and rotation exist in the spacetime. That is to say, mass, the Maxwell field and “rotation field” do no effect on the “entropy sum”.

3 “Entropy sum” of (A)dS black hole with other extra matter field

This section is devoted to the discussion about the “Entropy sum” of four dimensional (A)dS black hole with other extra matter field. The extra matter field of example we present here is the scalar field and with higher curvature terms. When there is scalar field in the spacetime, one will find that the “entropy sum” is dependent of the cosmological constant and the constant signifying the strength of self-interacting potential for the scalar field in both the conformally coupling frame and the minimally coupling frame. When we consider the charged rotating and static black holes in Einstein-Weyl theory, the “entropy sum” is shown a only dependence of the constant characterizing the strength of higher curvature terms, even the cosmological constant dependence is vanishing. For both cases, we conclude that the new “universal property”, “entropy sum” does not depend on the conserved charges M (mass), Q (from Maxwell field) and J (from rotation). One can believe that, “entropy sum” of all horizons (including the “virtual” horizon) does depend on those constants, which characterize the strength of the background fields.

3.1 Scalar hairy black holes

We consider the Einstein-Maxwell system in four dimensions with a cosmological constant Λ and a real conformally coupled self-interacting scalar field, described by the action

$$\mathcal{L} = \int d^4x \sqrt{-g} \left(\frac{R - 2\Lambda}{16\pi} - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{12} R \phi^2 - \alpha \phi^4 \right) - \frac{1}{16\pi} \int d^4x \sqrt{-g} F^{\mu\nu} F_{\mu\nu}, \quad (3.1)$$

where the parameter α is arbitrary self-interaction constant, which signify the coupling strength between gravity and the scalar field. The first well-known solution for this action is the “MTZ” black hole [20]. Here we will focus on the charged “MTZ” black hole solution with the metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \quad (3.2)$$

where $d\Omega^2$ is the line element of the 2-dimensional surface Σ

$$d\Omega^2 = \begin{cases} d\theta^2 + \sin^2 \theta d\varphi^2, & \text{sphere } S^2; \\ d\theta^2 + \theta^2 d\varphi^2, & \text{flat } \mathbb{R}^2; \\ d\theta^2 + \sinh^2 \theta d\varphi^2, & \text{hyperbolic } H^2. \end{cases} \quad (3.3)$$

The metric function is

$$f(r) = -\frac{\Lambda r^2}{3} + \gamma \left(1 + \frac{\mu}{r} \right)^2 \quad (3.4)$$

with the electromagnetic potential is given by

$$A = -\frac{q}{r} dt \quad (3.5)$$

The parameter γ denotes the normalized curvature constant of the 2-dimensional sub-manifold Σ . γ can take $+1, 0$ and -1 , corresponding to sphere S^2 , flat \mathbb{R}^2 and hyperbolic manifold H^2 . We will only consider $\gamma = \pm 1$, as the flat case corresponds to no black hole but naked singularity. The integration constants μ and q are not independent since they must satisfy

$$q^2 = \gamma G \mu^2 \left(1 + \frac{2\pi \Lambda G}{9\alpha} \right) \quad (3.6)$$

We will set $G = 1$ in what follows. On the other hand, μ and q are relate to the conserved charges: mass M and electric charge Q :

$$M = -\gamma \frac{\sigma}{4\pi} \mu \quad Q = \frac{\sigma}{4\pi} q \quad (3.7)$$

Where σ is the “unit” area of the 2-dimensional surface Σ

$$\sigma = \begin{cases} 4\pi, & \gamma = 1; \\ 4\pi(g-1), g \geq 2, & g \text{ is the genus, } \gamma = -1. \end{cases} \quad (3.8)$$

However, eq. (3.6) shows that M and Q are not independent in this spacetime. Thus, one can expect that the “entropy product” of all horizons is dependent on the conserved charges Q , which means a mass dependence as well. This a new failed example of the mass independence of “entropy product” relation.

There are two parameters $\Lambda = \pm \frac{3}{l^2}$ and $\gamma = \pm 1$ in the solution (3.4). They will corresponds to four solutions of the action. To give a brief check of the “entropy sum” relation, we will only present the discussions of two of these four solutions here. We focus on the dS black hole with sphere horizon ($\Lambda = \frac{3}{l^2}$ and $\gamma = 1$) [21] and the AdS black hole with hyperbolic horizon $\Lambda = -\frac{3}{l^2}$ and $\gamma = -1$ [22]. However, one may note the other two case correspond to no black hole but solutions with naked singularity. Then the metric function $f(r)$ (3.4) takes the form

$$f(r) = \mp \frac{r^2}{l^2} \pm \left(1 \mp \frac{M}{r}\right)^2 \quad (3.9)$$

where the upper and lower of sign stand for the dS and AdS solution respectively. There are four roots for this metric function [21, 23]

$$\begin{aligned} r_1 &= \frac{l}{2} \left(1 + \sqrt{1 \mp \frac{4M}{l}}\right) & r_2 &= \frac{l}{2} \left(1 - \sqrt{1 \mp \frac{4M}{l}}\right) \\ r_3 &= \frac{l}{2} \left(-1 + \sqrt{1 \pm \frac{4M}{l}}\right) & r_4 &= \frac{l}{2} \left(-1 - \sqrt{1 \pm \frac{4M}{l}}\right) \end{aligned}$$

Both black holes have possessed cosmological, event and inner horizons, given by the radial coordinate as r_1 , r_2 , r_3 respectively, and the r_4 is corresponding to a “virtual” horizon. One need note that, in fact we are considering the black holes with some special black hole mass M , in order to have multi-real roots, as we are interested in the “entropy product” of multi-horizons black hole.

The entropy corresponds to each horizon is

$$S(r_i) = \pi r_i^2 \left(1 - \frac{\phi(r_i)^2}{6}\right). \quad (3.10)$$

where the scalar field behaviours as [22]

$$\phi(r) = \sqrt{-\frac{\Lambda}{6\alpha}} \left(\frac{M}{r \mp M}\right) = \sqrt{\mp \frac{1}{2\alpha l^2}} \left(\frac{M}{r \mp M}\right) \quad (3.11)$$

Note α and Λ have opposite signs. After some direct calculation, we can find “entropy sum” of all four horizons

$$\sum_{i=1}^4 S(r_i) = 2\pi l^2 \pm \frac{\pi}{6\alpha} = \pm \frac{6\pi}{\Lambda} \pm \frac{\pi}{6\alpha} \quad (3.12)$$

To give the conclusion, when there is extra scalar field in the spacetime, one will find the “entropy sum” is dependent of the cosmological constant and the constant signifying the

coupling strength between gravity and the scalar field in the conformally coupling frame. It is also interesting to consider it in the minimally coupling frame. The result is shown in appendix A. We find the “entropy sum” is also dependent of the cosmological constant and the constant characterizing the strength of self-interacting potential of the scalar field. What we emphasize is the “virtual” horizon cannot be dropped, otherwise we cannot get the “entropy sum” result, which has background field constant dependence and conserved charge independence (Here is M -independence and Q -independence). In this sense, we say the “entropy sum” is “universal” in this theory. On the other hand, comparing this case with that of kerr-(A)dS black hole (see eq. (2.14)), it seems like that the topology of the sub-manifold Σ can modify the “entropy sum” in someway.

3.2 Charged rotating and static black holes in Einstein-Weyl theory

The dyonic black hole solution in $D = 4$ charged Einstein-Weyl theory has the Lagrangian [4, 24].

$$\begin{aligned}\mathcal{L} &= \sqrt{-g} \left(\frac{1}{2} \alpha C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} + \frac{1}{3} \alpha F^{\mu\nu} F_{\mu\nu} \right) \\ &= \sqrt{-g} \left(\alpha R^{\mu\nu} R_{\mu\nu} - \frac{1}{3} \alpha R^2 + \frac{1}{3} \alpha F^2 \right) + \alpha \mathcal{L}_{GB}.\end{aligned}\quad (3.13)$$

where \mathcal{L}_{GB} denotes the Gauss-Bonnet Lagrangian. And the charged rotating AdS black hole solution can be written as [4, 24]

$$ds_4^2 = \rho^2 \left(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta} \right) + \frac{\Delta_\theta \sin^2 \theta}{\rho^2} \left(a dt - (r^2 + a^2) \frac{d\phi}{\Xi} \right)^2 - \frac{\Delta_r}{\rho^2} \left(dt - a \sin^2 \theta \frac{d\phi}{\Xi} \right)^2, \quad (3.14)$$

where

$$\begin{aligned}\rho^2 &= r^2 + a^2 \cos^2 \theta & \Delta_\theta &= 1 - g^2 a^2 \cos^2 \theta \\ \Xi &= 1 - g^2 a^2 & \Delta_r &= (r^2 + a^2)(1 + g^2 r^2) - 2mr + \frac{(p^2 + q^2)r^3}{6m}\end{aligned}$$

where $\Lambda = -3g^2$ is the cosmological constant. In what follows we have set magnetic charge $p = 0$ so that there is only an electric charge q . Solve the equation $\Delta_r = 0$, one can obtain four horizons. The entropy on these horizons are Wald entropy which do not satisfy the area theorem, derived in [24] and have the form [4, 24].

$$S(r_i) = \frac{2\pi\alpha}{\Xi} \left(1 + g^2 r_i^2 + \frac{q^2 r_i}{6m} - c\Xi \right), \quad (3.15)$$

where the constant c is numerical and corresponds to adding a constant multiple of the Gauss-Bonnet invariant to the action [4]. Directly we calculate the sum of horizons entropy

$$\sum_{i=1}^4 S(r_i) = 4\pi\alpha(1 - 2c) \quad (3.16)$$

This result does not relate to the conserved quantities: total energy E , charge Q and angular momentum J [4, 24].

$$\begin{aligned} E &= \frac{2\alpha g^2}{\Xi^2} \left(m + \frac{a^2 q^2}{12m} \right) \\ Q &= \frac{\alpha q}{3\Xi} \\ J &= \frac{2a\alpha g^2}{\Xi^2} \left(m + \frac{q^2}{12mg^2} \right) \end{aligned}$$

Thus, the “entropy sum” is shown a only dependence of the constant α , even the cosmological constant dependence is vanishing.

Next we consider a special case, the charged static dS black hole solution i.e. $J = 0$, which the four horizons reduce to three [4]. Here $\Lambda = 3g^2$ is the cosmological constant. The metric of static black hole is (a detail analysis of general solution is given in [25])

$$\begin{aligned} ds^2 &= -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega_2^2 \\ A &= -\frac{q}{r} dt \end{aligned} \tag{3.17}$$

$$f = -\frac{\Lambda r^2}{3} + c_1 r + c_0 + \frac{d}{r} \tag{3.18}$$

$$3c_1 d + 1 + q^2 = c_0^2.$$

For the static black hole, the entropy for each horizon is [4]

$$S(r_i) = -\frac{2\pi\alpha(3d + (c_0 + 2)r_i)}{3r_i}. \tag{3.19}$$

We calculate the “entropy sum” to be

$$\sum_{i=1}^3 S(r_i) = -4\pi\alpha. \tag{3.20}$$

There are no conserved quantities E, Q in the “entropy sum” (3.16) and (3.20). So we could say that the “entropy sum” is shown a only dependence of the constant α , which characterizes the strength of higher curvature terms, even the cosmological constant dependence is vanishing.

4 Conclusion and discussion

In this paper, we find another “universal property” of entropy, the “entropy sum” relation of black holes in four dimensional (anti-)de-Sitter asymptotical background. We mainly test “entropy sum” relation in static, stationary black hole and some black holes with other extra matter source (scalar hair and higher curvature terms) in the asymptotical (anti-)de-sitter spacetime background. They depend only on the cosmological constant with the necessary effect of the un-physical “virtual” horizon included and in the spacetime only cosmological

constant, mass of the black hole, rotation parameter and the Maxwell field exist. When there is more extra matter field in the spacetime, one will find the “entropy sum” is also dependent of the strength of these extra matter field. For both cases, we conclude that the new “universal property”, that is, the “entropy sum” does not depend on the conserved charges: M (mass), Q (from Maxwell field) and J (from rotation), while it does depend on the property of background spacetime. To say it more accurately, it does depend on those constants, which characterize the strength of the background fields. When there is extra degree of freedom, that is the scalar field in the spacetime, it is dependent on the cosmological constant and the constant signifying the strength of self-interacting potential of the scalar field in both the conformally coupling frame and in the minimally coupling frame as shown in appendix A. Besides, in the Einstein-Maxwell-scalar-AdS spacetime, it seems like that the topology of the sub-manifold Σ can modify the “entropy sum” in some way; we also point out the mass independence of the “entropy product” relation failed in this case. When we consider the charged rotating and static black holes in the Einstein-Weyl theory, the “entropy sum” is shown to be only dependence on the constant characterizing the strength of higher curvature terms, even if the cosmological constant dependence is vanishing. What we emphasize is the “virtual” horizon cannot be dropped, otherwise we cannot get the “entropy sum” relation with the background field constant dependence. In this sense, we say the “entropy sum” is “universal” in the theory presented in this paper. One shall note that the “entropy sum” is negative in some black hole case, which maybe result from the effect of the entropy of the work out “virtual” horizon.

To give a whole look of the “entropy sum”, we finally consider it in the Kerr black hole case as shown in appendix B. We find only mass dependence of “entropy sum” appears. It makes us believe that, considering the dependence properties of “entropy sum” relation, the mass background field may be regarded as the next order of cosmological constant background field and extra matter field, while the Maxwell field and “rotation field” always play no role. Explaining the relationship between the “entropy sum” and background properties still are open problems, which is left to be a future work.

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A The “entropy sum” in the minimally coupling frame of Einstein-Maxwell-scalar-AdS spacetime

We consider the “entropy sum” in the minimally coupling frame of Einstein-Maxwell-scalar-AdS spacetime. One can obtain the solution directly by taking a conformal transformation of that in the conformally coupling frame [22, 26, 27]. Here we take the AdS black hole with hyperbolic horizon $\Lambda = -\frac{3}{l^2}$ and $\gamma = -1$ [22] in section 3.1 as an example. The

corresponding conformal transformation is

$$\Omega^2 = 1 - \frac{1}{6}\phi^2 = 1 - \frac{1}{12\alpha l^2} \left(\frac{M}{r+M} \right)^2. \quad (\text{A.1})$$

Then we introduce a new scalar field Φ

$$\tanh \left(\sqrt{\frac{1}{6}} \Phi \right) = \sqrt{\frac{1}{6}} \phi \quad (\text{A.2})$$

with $\Phi(r)$ behaviour as

$$\Phi(r) = \sqrt{6} \operatorname{arctanh} \left(\sqrt{\frac{1}{12\alpha l^2}} \frac{M}{r+M} \right) \quad (\text{A.3})$$

One can obtain the theory in the minimally coupling frame with the action

$$\mathcal{L} = \int d^4x \sqrt{-\hat{g}} \left(\frac{\hat{R}}{16\pi} - \frac{1}{2} \hat{g}^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - V(\Phi) \right) - \frac{1}{16\pi} \int d^4x \sqrt{-\hat{g}} F^{\mu\nu} F_{\mu\nu}, \quad (\text{A.4})$$

where the new self-interaction potential $V(\Phi)$ takes the form

$$V(\Phi) = \frac{\Lambda}{8\pi} \left(\cosh^4 \sqrt{\frac{1}{6}} \Phi + \frac{8\pi}{\Lambda} 36\alpha \sinh^4 \sqrt{\frac{1}{6}} \Phi \right) \quad (\text{A.5})$$

The transformed, minimal coupled version line element is

$$d\hat{s}^2 = \Omega^2 \left[- \left(\frac{r^2}{l^2} - \left(1 + \frac{M}{r} \right)^2 \right) dt^2 + \frac{dr^2}{\frac{r^2}{l^2} - \left(1 + \frac{M}{r} \right)^2} + r^2 d\sigma^2 \right] \quad (\text{A.6})$$

We still need to introduce a new radial coordinate $R^2 = r^2 \Omega^2$ to get the familiar coordinate frame, which one will obtain the usual Benkenstain-Hawking area entropy

$$S(R_i) = \frac{A(R_i)}{4} = \frac{\sigma R_i^2}{4} \quad (\text{A.7})$$

Then some calculation lead to

$$\begin{aligned} \sum_{i=1}^4 S_i &= \sum_{i=1}^4 S(R_i) = \sum_{i=1}^4 \frac{\sigma r_i^2}{4} \Omega^2 \\ &= \sum_{i=1}^4 \pi r_i^2 \left(1 - \frac{\phi(r_i)^2}{6} \right) \\ &= -\frac{6\pi}{\Lambda} - \frac{\pi}{6\alpha} \end{aligned}$$

which is the same as shown in section 3.1. Obviously the same rules of “entropy sum” still holds. When in the minimally coupling frame, “entropy sum” is also dependent of the cosmological constant and the constant characterizing the strength of self-interacting potential of the scalar field.

B The “entropy sum” of Kerr black hole

To give a whole look of the “entropy sum”, we consider it in asymptotical flat spacetime without any matter source, taking Kerr black hole [28] as an example. All horizons of Kerr black hole are

$$\begin{aligned} r_1 &= M + \sqrt{M^2 - a^2} \\ r_2 &= M - \sqrt{M^2 - a^2} \end{aligned}$$

and area of each horizon are

$$A(r_i) = 4\pi(r_i^2 + a^2)$$

entropy respectively

$$S(r_i) = \frac{A(r_i)}{4} \quad (\text{B.1})$$

then “entropy sum” is

$$\sum_{i=1}^2 S(r_i) = 4\pi M^2 \quad (\text{B.2})$$

apparently it is mass dependent. It seems that, considering the dependence of “entropy sum”, the mass background field maybe is the next order of cosmological constant background field and extra matter field, while the Maxwell field and “rotation field” always play no role.

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